VALSCALC

Chapter 2 Free-Response Practice Test

Directions: This practice test features free-response questions based on the content in Chapter 2: Differentiation Rules.

2.1: Defining a Derivative
2.2: Differentiating Power, Exponential, and Sinusoidal Functions
2.3: Product Rule and Quotient Rule
2.4: Chain Rule
2.5: Implicit Differentiation and Differentiating Inverse Functions
2.6: Differentiating Logarithmic Functions
2.7: Related Rates
2.8: Linearization and Differentials
2.9: Hyperbolic Functions

For each question, show your work. If you encounter difficulties with a question, then move on and return to it later. Follow these guidelines:

- Do not use a calculator of any kind. All of these problems are designed to contain simple numbers.
- Adhere to the time limit of 90 minutes.
- After you complete all the questions, score yourself according to the Solutions document. Note any topics that require revision.

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Differentiation Rules

Number of Questions—22

Suggested Time—1 hour 30 minutes

NO CALCULATOR

Scoring Chart

Section	Points Earned	Points Available
Rapid Derivatives		20
Short Questions		50
Question 21		15
Question 22		15
TOTAL		**100**

Rapid Derivatives

Find the derivative of each function. Simplify your answers wherever possible.

1. $y = \sqrt[6]{x}$ *(2 pts.)*

2. $y = e^x \sec x$ *(2 pts.)*

3. $y = x\tan^{-1} 2x$ *(2 pts.)*

4. $y = \log_7 x$ *(2 pts.)*

5. $y = \dfrac{\cot x}{x^3 + x}$ *(2 pts.)*

6. $y = \cos\left(\dfrac{2}{x^3}\right)$ *(2 pts.)*

7. $y = \ln(x^4 - 6x^2 + 5)$ *(2 pts.)*

8. $y = \tan^{-1}(\operatorname{sech} x)$ *(2 pts.)*

9. $y = \sin^2\left(\sqrt{9-x^2}\right)$ *(2 pts.)*

10. $y = x\left(4-e^{-x/2}\right)$ *(2 pts.)*

Short Questions

11. Let $f(x) = x^2 + x - 3$. Use the limit definition of a derivative to find $f'(1)$. (*5 pts.*)

12. If f is a continuous function such that $f(1) = 5$ and $f'(1) = 2$, then use linearization to approximate $f(1.2)$. (*5 pts.*)

13. For $xy^3 + y^2 = 8$, find $\frac{\mathrm{d}y}{\mathrm{d}x}$. *(5 pts.)*

14. Using differentials, estimate the amount by which a cube's volume increases as its side lengths increase from 5 inches to 5.1 inches. *(5 pts.)*

15. Prove that $\cosh 3x - \sinh 3x = e^{-3x}$. *(5 pts.)*

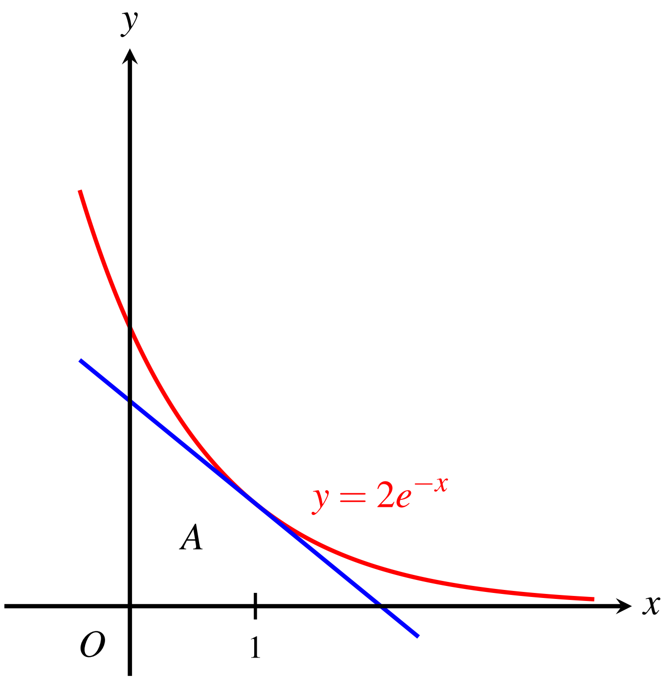
16. Write an equation of the line normal to the curve $y = x^5 - 2x^3 - 4$ at $x = 1$. *(5 pts.)*

17. If $g(x) = 3x^5 - x$, then calculate $\left(g^{-1}\right)'(2)$. (5 pts.)

18. A uniform tank has a cross-sectional area of 100 square centimeters. Water is pumped into the tank at a rate of 7 cubic centimeters per minute. How quickly is the tank's water level rising? (5 pts.)

19. Use Logarithmic Differentiation to find $\dfrac{\mathrm{d}}{\mathrm{d}x}\sqrt[3]{\dfrac{x^3\sin 2x}{x^2-4}}$, assuming the function is positive. *(5 pts.)*

20. The triangle A is bounded by the x-axis, the y-axis, and the line tangent to $y = 2e^{-x}$ at $x = 1$, as shown in the figure. Calculate the area of A. *(5 pts.)*



Long Questions

21. The graph C is given by the implicit equation $3x^2 + y^2 = 2 - y$.

(a) Show that $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{-6x}{2y+1}$. *(3 pts.)*

(b) Write an equation of the line tangent to C at the point $(1,1)$. *(2 pts.)*

(c) Find all the points at which C has horizontal tangents. *(3 pts.)*

(d) Where does C have vertical tangents? *(3 pts.)*

(e) Find $\frac{d^2y}{dx^2}$ in terms of x and y. *(4 pts.)*

22. Particle P travels along the x-axis as modeled by the position function $x(t) = t^3 - 2t^2 + 3t$. Particle Q moves along the y-axis according to the position function $y(t) = 2t + 6$. Both $x(t)$ and $y(t)$ are measured in feet, and $t \geqslant 0$ is measured in seconds.

(a) At the moment when $t = 2$, how far apart are particles P and Q ? *(1 pt.)*

(b) Calculate both particles' velocities when $t = 2$. *(4 pts.)*

(c) When $t = 2$, calculate the rate at which the distance between particles P and Q is changing. *(6 pts.)*

(d) Find particle P's acceleration when $t = 2$. *(4 pts.)*

This marks the end of the test. The solutions and scoring rubric begin on the next page.

Rapid Derivatives (2 points each)

1. Because $\sqrt[6]{x} = x^{1/6}$, the Power Rule gives

$$\frac{dy}{dx} = \tfrac{1}{6}x^{-5/6} = \boxed{\frac{1}{6\sqrt[6]{x^5}}}$$ **

2. The Product Rule gives

$$\frac{dy}{dx} = \boxed{e^x \sec x \tan x + e^x \sec x}$$ **

3. By the Chain Rule,

$$\frac{d}{dx}\left(\tan^{-1} 2x\right) = \frac{2}{(2x)^2+1} = \frac{2}{4x^2+1}.$$

Thus, the Product Rule gives

$$\frac{dy}{dx} = x\left(\frac{2}{4x^2+1}\right) + (1)\tan^{-1} 2x = \boxed{\frac{2x}{4x^2+1} + \tan^{-1} 2x}$$ **

4. By the Change of Base Formula for Logarithms, $\log_7 x = \dfrac{\ln x}{\ln 7}$. Accordingly,

$$\frac{dy}{dx} = \frac{d}{dx}\left(\frac{\ln x}{\ln 7}\right) = \boxed{\frac{1}{x \ln 7}}$$ **

5. By the Quotient Rule,

$$\frac{dy}{dx} = \frac{(-\csc^2 x)(x^3+x) - \cot x(3x^2+1)}{(x^3+x)^2} = \boxed{\frac{-x^3\csc^2 x - x\csc^2 x - 3x^2\cot x - \cot x}{x^6+2x^4+x^2}}$$ **

6. Because $2/x^3 = 2x^{-3}$, the Power Rule yields

$$\frac{d}{dx}\left(\frac{2}{x^3}\right) = -6x^{-4} = -\frac{6}{x^4}.$$

Then the Chain Rule gives

$$\frac{dy}{dx} = -\sin\left(\frac{2}{x^3}\right)\left(-\frac{6}{x^4}\right) = \boxed{\frac{6}{x^4}\sin\left(\frac{2}{x^3}\right)}$$ **

7. By the Chain Rule,

$$\frac{dy}{dx} = \frac{1}{x^4-6x^2+5}(4x^3-12x) = \boxed{\frac{4x^3-12x}{x^4-6x^2+5}}$$ **

8. By the Chain Rule,

$$\frac{dy}{dx} = \frac{1}{1+\operatorname{sech}^2 x}(-\operatorname{sech} x\tanh x) = \boxed{-\frac{\operatorname{sech} x\tanh x}{1+\operatorname{sech}^2 x}}$$ **

9. By the Chain Rule,

$$\frac{dy}{dx} = 2\sin\left(\sqrt{9-x^2}\right)\cos\left(\sqrt{9-x^2}\right)\left(\frac{-2x}{2\sqrt{9-x^2}}\right) = \boxed{\frac{-2x}{\sqrt{9-x^2}}\sin\left(\sqrt{9-x^2}\right)\cos\left(\sqrt{9-x^2}\right)}$$ **

10. By the Product Rule,

$$\frac{dy}{dx} = (1)\left(4-e^{-x/2}\right)+x\left(\tfrac{1}{2}e^{-x/2}\right) = \boxed{4-e^{-x/2}+\tfrac{1}{2}xe^{-x/2}}$$ **

Short Questions (5 points each)

11. By the limit definition of a derivative,

$$f'(1) = \lim_{h \to 0} \frac{f(1+h) - f(1)}{h}\,. \qquad *$$

Note that $f(1) = -1$ and

$$f(1+h) = (1+h)^2 + (1+h) - 3 = 3h + h^2 - 1\,. \qquad *$$

So the limit becomes

$$\begin{aligned} f'(1) &= \lim_{h \to 0} \frac{(3h + h^2 - 1) - (-1)}{h} && * \\ &= \lim_{h \to 0} \frac{3h + h^2}{h} && * \\ &= \lim_{h \to 0} (3 + h) \\ &= \boxed{3} && * \end{aligned}$$

12. We have

$$\begin{aligned} f(1.2) &\approx f(1) + f'(1)(x - 1) && ** \\ &= 5 + 2(1.2 - 1) && * \\ &= \boxed{5.4} && ** \end{aligned}$$

13. Differentiating both sides gives

$$y^3 + 3xy^2 \frac{\mathrm{d}y}{\mathrm{d}x} + 2y\frac{\mathrm{d}y}{\mathrm{d}x} = 0 \quad **$$

$$\frac{\mathrm{d}y}{\mathrm{d}x}(3xy^2 + 2y) = -y^3 \quad *$$

$$\implies \frac{\mathrm{d}y}{\mathrm{d}x} = \boxed{-\frac{y^2}{3xy+2}} \quad **$$

14. The cube's volume V is given by $V = s^3$, where s is a side length. The differential of V is

$$\mathrm{d}V = 3s^2\,\mathrm{d}s\,. \quad *$$

Taking $\mathrm{d}V \approx \Delta V$ and $\mathrm{d}s = \Delta s = 0.1$ shows

$$\Delta V \approx 3(5)^2(0.1) \quad **$$

$$= \boxed{7.5} \quad **$$

15. We have

$$\cosh 3x = \frac{e^{3x}+e^{-3x}}{2} \qquad \text{and} \qquad \sinh 3x = \frac{e^{3x}-e^{-3x}}{2}\,. \quad **$$

Then

$$\cosh 3x - \sinh 3x = \frac{e^{3x}+e^{-3x}}{2} - \frac{e^{3x}-e^{-3x}}{2}$$

$$= \frac{2e^{-3x}}{2} \quad *$$

$$= \boxed{e^{-3x}} \quad **$$

16. We have

$$\left.\frac{\mathrm{d}y}{\mathrm{d}x}\right|_{x=1} = \left.(5x^4 - 6x^2)\right|_{x=1} = -1\,. \quad *$$

So the slope of the normal line is

$$-\frac{1}{(-1)} = 1\,.$$ *

A point on the curve is $(1,-5)$, so an equation of the normal line is

$$y-(-5) = 1(x-1)$$ *

$$\boxed{y = x-6}$$ **

17. Because $g(1) = 2$, we have

$$g^{-1}(2) = 1\,.$$ *

Also,

$$g'(1) = (15x^4 - 1)\Big|_{x=1} = 14\,.$$ *

So

$$\left(g^{-1}\right)'(2) = \frac{1}{g'(g(^{-1}(2))}$$ *

$$= \frac{1}{g'(1)}$$

$$= \boxed{\frac{1}{14}}$$ **

18. Let V be the tank's volume and h be its water level. Since $V = 100h$ and $\mathrm{d}V/\mathrm{d}t = 7$, we have

$$\frac{\mathrm{d}V}{\mathrm{d}t} = 100\,\frac{\mathrm{d}h}{\mathrm{d}t}$$ *

$$7 = 100\,\frac{\mathrm{d}h}{\mathrm{d}t}$$ *

$$\Longrightarrow \frac{\mathrm{d}h}{\mathrm{d}t} = \boxed{\frac{7}{100}\text{ cm/min}}$$ ***

19. With $y = \sqrt[3]{\dfrac{x^3 \sin 2x}{x^2 - 4}}$, it follows that

$$\begin{aligned}
\ln y &= \tfrac{1}{3} \ln \left(\frac{x^3 \sin 2x}{x^2 - 4} \right) && * \\
&= \tfrac{1}{3} \left[\ln(x^3) + \ln(\sin 2x) - \ln(x^2 - 4) \right] \\
&= \tfrac{1}{3} \left[3 \ln x + \ln(\sin 2x) - \ln(x^2 - 4) \right] \\
&= \ln x + \tfrac{1}{3} \ln(\sin 2x) - \tfrac{1}{3} \ln(x^2 - 4)\,. && *
\end{aligned}$$

Performing Implicit Differentiation produces

$$\begin{aligned}
\frac{1}{y} \frac{dy}{dx} &= \frac{1}{x} + \frac{2\cos 2x}{3 \sin 2x} - \frac{2x}{3x^2 - 12} && * \\
\frac{1}{\sqrt[3]{\dfrac{x^3 \sin 2x}{x^2 - 4}}} \frac{dy}{dx} &= \frac{1}{x} + \frac{2\cos 2x}{3 \sin 2x} - \frac{2x}{3x^2 - 12} && * \\
\implies \frac{dy}{dx} &= \sqrt[3]{\frac{x^3 \sin 2x}{x^2 - 4}} \left(\frac{1}{x} + \frac{2\cos 2x}{3 \sin 2x} - \frac{2x}{3x^2 - 12} \right) \\
&= \boxed{\sqrt[3]{\frac{x^3 \sin 2x}{x^2 - 4}} \left(\frac{1}{x} + \frac{2}{3} \cot 2x - \frac{2x}{3x^2 - 12} \right)} && *
\end{aligned}$$

20. A point on the graph is $(1, 2/e)$. Its slope at $x = 1$ is

$$\left. \frac{dy}{dx} \right|_{x=1} = \left. -2e^{-x} \right|_{x=1} = -\frac{2}{e}\,.$$

Consequently, an equation of the tangent line is

$$\begin{aligned}
y - \frac{2}{e} &= -\frac{2}{e}(x - 1) \\
y &= -\frac{2}{e} x + \frac{4}{e}\,. && *
\end{aligned}$$

When the line hits the x-axis, $y = 0$ and so

$$0 = -\frac{2}{e}x + \frac{4}{e}$$
$$\implies x = 2\,. \qquad *$$

When the line hits the y-axis, $x = 0$ and so

$$y = -\frac{2}{e}(0) + \frac{4}{e} = \frac{4}{e}\,. \qquad *$$

Hence, the triangle's area is

$$\begin{aligned} A &= \tfrac{1}{2}(\text{base})(\text{height}) \\ &= \tfrac{1}{2}(2)\left(\frac{4}{e}\right) \\ &= \boxed{\frac{4}{e}} \qquad ** \end{aligned}$$

Long Questions (15 points each)

21. (a) Differentiating both sides of the implicit equation yields

$$6x + 2y\frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{\mathrm{d}y}{\mathrm{d}x} \qquad *$$

$$(2y+1)\frac{\mathrm{d}y}{\mathrm{d}x} = -6x \qquad *$$

$$\implies \frac{\mathrm{d}y}{\mathrm{d}x} = \boxed{\frac{-6x}{2y+1}} \qquad *$$

(b) At $(1,1)$, the slope is

$$\left.\frac{\mathrm{d}y}{\mathrm{d}x}\right|_{(1,1)} = \frac{-6(1)}{2(1)+1} = -2\,. \qquad *$$

Then an equation of the tangent line is

$$y - 1 = -2(x-1) \qquad \text{or} \qquad \boxed{y = -2x+3} \qquad *$$

(c) The curve C has horizontal tangents when $\frac{\mathrm{d}y}{\mathrm{d}x} = 0$. Solving shows

$$\frac{-6x}{2y+1} = 0$$

$$\implies x = 0\,. \qquad *$$

Substituting $x = 0$ into the implicit equation gives

$$3(0)^2 + y^2 = 2 - y \qquad *$$

$$y^2 + y - 2 = 0$$

$$(y+2)(y-1) = 0$$

$$\implies y = -2, 1\,.$$

Thus, C has horizontal tangents at

$$\boxed{(0,-2)} \qquad \text{and} \qquad \boxed{(0,1)}$$

(d) The curve C has vertical tangents when $\frac{dy}{dx}$ has a 0 denominator:

$$\begin{aligned} 2y+1 &= 0 \\ \implies y &= -\tfrac{1}{2}. \end{aligned}$$ *

Substituting $y = -\frac{1}{2}$ into the implicit equation shows

$$\begin{aligned} 3x^2 + \left(-\frac{1}{2}\right)^2 &= 2 - \left(-\frac{1}{2}\right) \\ 3x^2 &= \frac{9}{4} \\ x^2 &= \frac{3}{4} \\ \implies x &= -\frac{\sqrt{3}}{2}, \frac{\sqrt{3}}{2}. \end{aligned}$$ *

Thus, vertical tangents are located at

$$\boxed{\left(-\frac{\sqrt{3}}{2}, -\frac{1}{2}\right)} \quad \text{and} \quad \boxed{\left(\frac{\sqrt{3}}{2}, -\frac{1}{2}\right)}$$

(e) By the Quotient Rule,

$$\begin{aligned} \frac{d^2y}{dx^2} &= \frac{d}{dx}\left(\frac{dy}{dx}\right) \\ &= \frac{-6(2y+1) + 6x\left(2\frac{dy}{dx}\right)}{(2y+1)^2} \quad * \\ &= \frac{-6(2y+1) + 12x\left(\frac{-6x}{2y+1}\right)}{(2y+1)^2} \quad * \\ &= \boxed{\frac{-6(2y+1)^2 - 72x^2}{(2y+1)^3}} \quad ** \end{aligned}$$

22. (a) When $t = 2$,

$$x(2) = (2)^3 - 2(2)^2 + 3(2) = 6$$

$$y(2) = 2(2) + 6 = 10\,.$$

Then the distance is

$$r(2) = \sqrt{6^2 + 10^2} = \boxed{\sqrt{136}\text{ ft}}$$ *

(b) Particle P's velocity function is

$$x'(t) = 3t^2 - 4t + 3\,.$$ *

Its velocity at $t = 2$ is

$$x'(2) = 3(2)^2 - 4(2) + 3 = \boxed{7\text{ ft/sec}}$$ *

Particle Q's velocity function is

$$y'(t) = 2\,.$$ *

Hence,

$$y'(2) = \boxed{2\text{ ft/sec}}$$ *

(c) If r is the distance between P and Q, then the Pythagorean Theorem gives

$$r^2 = x^2 + y^2\,.$$

Differentiating both sides with respect to time yields

$$2r\frac{\mathrm{d}r}{\mathrm{d}t} = 2x\frac{\mathrm{d}x}{\mathrm{d}t} + 2y\frac{\mathrm{d}y}{\mathrm{d}t}\,.$$ **

When $t = 2$, the distances from part (a)—namely, $x = 6$, $y = 10$, and $r = \sqrt{136}$—govern. Similarly, from part (b), $\mathrm{d}x/\mathrm{d}t = 7$ and $\mathrm{d}y/\mathrm{d}t = 2$. Substituting these values gives

$$2\sqrt{136}\,\frac{\mathrm{d}r}{\mathrm{d}t} = 2(6)(7) + 2(10)(2)$$ *

$$\implies \frac{\mathrm{d}r}{\mathrm{d}t} = \boxed{\frac{62}{\sqrt{136}}\text{ ft/sec}}$$ ***

(d) Particle P's acceleration function is $x''(t)$. We have

$$x''(t) = 6t - 4\,.$$ **

Thus, at $t = 2$

$$x''(2) = 6(2) - 4 = \boxed{8 \text{ ft/sec}^2}$$

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